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ABSENCE OF HIGHER DERIVATIVES IN THE RENORMALIZATION OF PROPAGATORS IN QUANTUM FIELD THEORIES WITH INFINITELY MANY COUPLINGS

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Abstract

I study some aspects of the renormalization of quantum field theories with infinitely many couplings in arbitrary space-time dimensions. I prove that when the space-time manifold admits a metric of constant curvature the propagator is not affected by terms with higher derivatives. More generally, certain lagrangian terms are not turned on by renormalization, if they are absent at the tree level. This restricts the form of the action of a non-renormalizable theory, and has applications to quantum gravity. The new action contains infinitely many couplings, but not all of the ones that might have been expected. In quantum gravity, the metric of constant curvature is an extremal, but not a minimum, of the complete action. Nonetheless, it appears to be the right perturbative vacuum, at least when the curvature is negative, suggesting that the quantum vacuum has a negative asymptotically constant curvature. The results of this paper give also a set of rules for a more economical use of effective quantum field theories and suggest that it might be possible to give mathematical sense to theories with infinitely many couplings at high energies, to search for physical predictions.

1 Introduction

The quantization of gravity is still elusive. The removal of divergences of quantum gravity is possible only in the presence of infinitely many independent coupling constants and it is hard, although not impossible in principle, to find computable quantities and make physical predictions about the high-energy behavior of the theory. On the other hand, the strenuous efforts spent in the recent years to search for approaches "beyond quantum field theory" have not produced significant breakthroughs. In the absence of convincing alternatives, it is wiser to investigate the problem of quantum gravity in the best established framework and get used to work with infinitely many couplings.

The action

$$-\frac{1}{\kappa^2}\int\sqrt{g}\ \ (R-\Lambda) \tag{1.1}$$

is not positive definite, in the Euclidean framework. The complete action of quantum gravity, however, is not (1.1), but contains infinitely many terms besides (1.1), with arbitrarily high powers and derivatives of the curvature tensors. In these circumstances, we need not worry about the positive indefiniteness of (1.1). We can define the functional integral formally, and perturbatively, relaxing the requirement of rigorous convergence of the term-by-term functional integration.

Usually, non-renormalizable theories are used as effective low-energy theories, because the number of parameters that are necessary to remove the divergences is finite at low energies (but grows with the energy).

As opposed to an "effective" theory, a "fundamental theory" should be well-defined and predictive at arbitrarily high energies. In the usual sense, a fundamental theory is predictive only if it contains a finite number of independent coupling constants. If it contains infinitely many coupling constants, certain subclasses of correlation functions might still depend only on a finite number of parameters, or a finite number of functions of them. Then the problem is to identify such classes of correlation functions.

In this paper I consider the theories with infinitely many coupling constants and show that some questions are well posed, and can be answered. The purpose of this research is to construct a class of theories that lie at an intermediate stage between effective and fundamental quantum field theory, e.g. quantum gravity with infinitely many parameters. The final goal is to use these theories to derive *some* physical predictions beyond the low-energy domain and possibly classify *which* physical predictions can be derived from these theories.

The first step in the task of giving mathematical sense to quantum field theories with infinitely many parameters at arbitrary energies is to show that a unitary propagator is not driven by renormalization into a non-unitary (typically, higher-derivative) propagator. The reason to worry that this might happen is that the non-renormalizability of the theory can potentially generate all sorts of counterterms, including those that can affect the propagator with undesirable higher derivatives. In this paper I show that this problem can be solved when the space-time manifold admits a metric of constant curvature. Furthermore, it is possible to screen the terms of the lagrangian and prove that, for example, a whole class of terms is not turned on by renormalization, if it is absent at the tree level.

In the absence of matter, quantum gravity is one-loop finite [1]: the one-loop divergent terms of the form R^2, $R_{\mu\nu}^2$ can be removed with a redefinition of the metric tensor. However, when gravity is coupled to matter, the counterterms R^2, $R_{\mu\nu}^2$ are non-trivial and can in principle be responsible for the appearance of unphysical singularities in the graviton propagator. The unphysical singularities of the propagator are relevant only at high energies and can be ignored if the perturbative expansion is truncated to a finite power of the energy [2], e.g. in the framework of effective field theory. However, this is a way to ignore the problem, rather than solve it. The final goal of perturbation theory is the resummation of the series expansion, at least in suitable correlation functions and physical quantities. Fortunately, in the presence of matter, the undesired counterterms R^2, $R_{\mu\nu}^2$ can be traded for renormalizations of the vertex couplings and a redefinition of the metric tensor, namely there exists a subtraction scheme where the graviton propagator is not affected by higher derivatives. Moreover, this fact generalizes to every order in the perturbative expansion and in arbitrary space-time dimensions, under some mild assumptions. The absence of unphysical singularities in the propagators is a necessary condition for unitarity, although not sufficient. It is worth saying that this result does not ensure that, for example, the two-point functions of the fields are free of Landau poles (Landau poles are difficult to treat also in the context of renormalizable theories).

To fix some basic terminology, I distinguish two subclasses of counterterms: the quadratic counterterms and the vertex counterterms. By quadratic counterterms I mean the counterterms that, expanded perturbatively, have contributions quadratic in the quantum fluctuations of the fields. By vertex counterterms I mean the counterterms that, expanded perturbatively, have no contributions quadratic and linear in the quantum fluctuations of the fields. Let us consider four-dimensional pure quantum gravity without a cosmological constant in detail,

$$\mathcal{L} = -\frac{1}{\kappa^2}\sqrt{g}R(x). \tag{1.2}$$

The one-loop divergences [1]

$$\frac{1}{8\pi^2\varepsilon}\sqrt{g}\left(\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}\right) \tag{1.3}$$

can be removed in two different ways.

i) New coupling constants are added and the lagrangian (1.2) is modified into

$$\mathcal{L}' = \frac{1}{\kappa^2}\sqrt{g}\left(-R(x) + \alpha_{\rm B}R^2 + \beta_{\rm B}R_{\mu\nu}R^{\mu\nu}\right), \tag{1.4}$$

The divergences (1.3) are removed defining the bare parameters

$$\alpha_{\rm B} = \left(\alpha - \frac{1}{8\pi^2\varepsilon}\frac{1}{120}\right)\mu^{-\varepsilon}, \qquad \beta_{\rm B} = \left(\beta - \frac{1}{8\pi^2\varepsilon}\frac{7}{20}\right)\mu^{-\varepsilon}.$$

The replacement (1.2) $\rightarrow$ (1.4) changes the theory into higher-derivative quantum gravity, which is renormalizable, but not unitary [3]. The renormalizability is due to the resummation of the power series of α and β in the graviton propagator. The unphysical singularities are of orders $1/\alpha$ and $1/\beta$.

Once the couplings α and β are introduced, there are energy domains where the values of α and β cannot be considered small, and the modified theory (1.4) of quantum gravity, as a fundamental theory, violates unitarity at high energies.

ii) We observe that (1.3) vanish using the field equations of (1.2) and remove (1.3) with field redefinitions of the form

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \frac{\kappa^2}{8\pi^2\varepsilon}\frac{1}{20}\left(-7R_{\mu\nu} + \frac{11}{3}g_{\mu\nu}R\right), \tag{1.5}$$

up to two- and higher-loop contributions. The theory remains non-renormalizable, but unitary. In this case, the couplings α and β are called "inessential" [2].

The explicit computation of the one-loop divergences is actually unnecessary to prove finiteness, because the one-loop divergences are a linear combination of terms quadratic in the curvature tensors, namely

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \qquad R_{\mu\nu}R^{\mu\nu}, \qquad R^2, \tag{1.6}$$

but the identity

$$\sqrt{g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) = \text{total derivative}, \tag{1.7}$$

can be used to convert $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ into the sum $4R_{\mu\nu}R^{\mu\nu} - R^2$, which is proportional to the vacuum field equations and can be eliminated with a redefinition of the metric tensor. However, (1.7) is true only in four dimensions, because to prove (1.7) it is necessary to use that a completely antisymmetric tensor with more than four indices vanishes. It is therefore natural to wonder whether in higher dimensions gravity is always driven to higher-derivative gravity by renormalization. I am going to show that this is not the case, because suitable generalizations of (1.7) do exist.

Finiteness of one-loop pure quantum gravity is a coincidence, spoiled by the presence of matter [1]. Moreover, Goroff and Sagnotti [4] proved that pure quantum gravity is not finite at the second loop order, but there appears a counterterm proportional to

$$R_{\mu\nu}^{\rho\sigma}R_{\alpha\beta}^{\mu\nu}R_{\rho\sigma}^{\alpha\beta}, \tag{1.8}$$

which cannot be reabsorbed by means of field redefinitions. Finally, in higher dimensions pure quantum gravity is not even finite at the one-loop order.

As anticipated above, the identity (1.7) admits a number of generalizations. If the metric is expanded around a flat background,

$$g_{\mu\nu} = \delta_{\mu\nu} + \phi_{\mu\nu},$$

the dimension-independent formula

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu\phi_{\mu\sigma} - \partial_\rho\partial_\mu\phi_{\nu\sigma} - \partial_\sigma\partial_\nu\phi_{\mu\rho} + \partial_\sigma\partial_\mu\phi_{\nu\rho}\right) + \mathcal{O}(\phi^2)$$

implies that in the combination

$$\sqrt{g}\ \mathrm{G} \equiv \sqrt{g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

the sum of the quadratic terms in $\phi_{\mu\nu}$ is a total derivative in every space-time dimension. This ensures that the integral $\int\sqrt{g}\ \mathrm{G}$ is a vertex,

$$\int\sqrt{g}\ \mathrm{G} = \mathcal{O}(\phi^3), \tag{1.9}$$

and therefore the divergences of the form (1.6) do not affect the propagator with higher derivatives. It is easy to prove that the right-hand side of (1.9) does not vanish in dimensions greater than four.

This observation can be generalized and applied to show that renormalization protects unitarity even in the presence of infinitely many couplings. I prove that it is possible to remove the higher-derivative quadratic counterterms by means of field redefinitions in two cases: in the absence of a cosmological term and when the space-time manifold admits a metric of constant curvature. This property holds also in the presence of matter and in arbitrary dimension greater than two, for theories containing fields of spin 0, 1/2, 1, 3/2 and 2.

I believe that the true meaning of the identity (1.7) and its generalizations, such as (1.9) and (7.37), is the absence of higher-derivative corrections to the propagators in theories with infinitely many couplings, rather than the finiteness of special truncations of some theories.

When the space-time manifold does not admit a metric of constant curvature, a more general version of the theorem ensures that a certain class of terms is not turned on by renormalization, if it is absent at the tree level. This allows us to write an explicit form of the lagrangian of quantum gravity, which does contain infinitely many parameters, but not all of the ones that we might have expected.

If the metric is expanded around a vacuum metric with non-constant curvature, the graviton propagator does contain higher derivatives. This happens also in non-renormalizable theories of fields with spin 0, 1/2 and 1, expanded around non-trivial backgrounds, for example istantons. The issue of unitarity around nontrivial backgrounds is delicate and, in the case of instantons, the matter is further complicated by the integration over the instanton moduli space. Here I do not prove sufficient conditions for unitarity, but a simple theorem ensuring that under the mentioned assumptions renormalization does not generate unphysical poles in the perturbative propagator.

I also show that the results of this paper priviledge a spacetime manifold admitting a metric of constant curvature and therefore suggest that the quantum vacuum has an asymptotically constant curvature. The metric of constant curvature is an extremal, although not a minimum of the complete action. Nevertheless, it seems to be the correct perturbative vacuum for the calculations in quantum gravity, at least when the constant curvature is negative.

The paper is organized as follows. In section 2 I briefly discuss the relation between higher derivatives, unphysical singularities of the propagator and normalization conditions of the couplings in quantum field theory. In section 4 I treat the non-renormalizable theories of fields with spin 0, 1/2 and 1, while in section 5 I treat the spin-3/2 fields. In sections 6 and 7 I study quantum gravity without and with a cosmological constant, respectively, and write the complete action of quantum gravity. In section 8 I make some observations about the perturbative vacuum and the vacuum of quantum gravity. Section 9 collects some conclusions.

2 Higher derivatives and unitarity

When the quadratic part of a lagrangian contains higher derivatives the initial conditions in classical mechanics and the normalization conditions for the two-point function in quantum field theory are non-standard and reveal the presence of ghosts.

Classical mechanics. In classical mechanics the presence of higher (time) derivatives implies that the solution of the equations of motion is not uniquely determined by the initial positions and velocities of the particles. For example, the lagrangian

$$\mathcal{L}=\frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2+\frac{\alpha}{2}\left(\frac{\mathrm{d}^2q}{\mathrm{d}t^2}\right)^2-V(q) \tag{2.10}$$

generates an equation of motion containing the fourth time derivative of q,

$$m\frac{\mathrm{d}^2q}{\mathrm{d}t^2}-\alpha\frac{\mathrm{d}^4q}{\mathrm{d}t^4}+\frac{\partial V(q)}{\partial q}=0, \tag{2.11}$$

whose solution is unique if the position, velocity, acceleration and the derivative of the acceleration at a reference time t_0 are specified:

$$q(t_0)=q_0,\qquad \frac{\mathrm{d}q}{\mathrm{d}t}(t_0)=v_0,\qquad \frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t_0)=a_0,\qquad \frac{\mathrm{d}^3q}{\mathrm{d}t^3}(t_0)=\dot{a}_0.$$

This means that the theory does not propagate just one field, but more fields. However, the additional fields are ghosts. This can be quickly viewed as follows.

The potential and kinetic terms of (2.10) are positive definite, if $\alpha>0$ and V is positive. If we introduce an additional variable Q and rewrite the lagrangian without higher derivatives, for example

$$\mathcal{L}'=\frac{m}{2}\left(\frac{\mathrm{d}\widetilde{q}}{\mathrm{d}t}\right)^2-\frac{\alpha^2}{2m}\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2-\frac{\alpha}{2}Q^2-V\left(\widetilde{q}+\frac{\alpha}{m}Q\right),$$

we obtain a theory whose equations of motion are equivalent to those of (2.11), with $\widetilde{q}=q-\alpha Q/m$. The new potential is still positive definite, but the new kinetic term is not positive definite.

Quantum field theory. Consider for example the φ^4 theory

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu\varphi\right)^2+\frac{m^2}{2}\varphi^2+\frac{\lambda}{4!}\varphi^4. \tag{2.12}$$

The physical constants and the normalization of the field φ have to be fixed at a reference energy scale μ by means of suitable *normalization conditions*, such as

$$\left.\frac{\partial^2\Gamma^{(2)}[p]}{\partial p^2}\right|_{p^2=\mu^2} = 1, \tag{2.13}$$

$$\left.\Gamma^{(2)}[p]\right|_{p^2=\mu^2} = \mu^2 + m^2, \qquad \left.\Gamma^{(4)}[p_1, p_2, p_3]\right|_S = \lambda. \tag{2.14}$$

where S denotes the symmetric condition $p_1^2 = p_2^2 = p_3^2 = \mu^2$, $s = t = u = 4\mu^2/3$, and $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_2 + p_3)^2$, as usual. Here m and λ denote the renormalized parameters, but might not be the physical mass and the physical coupling constant. The first condition (2.13) is conventional and fixes the normalization of the field φ, which has no physical significance. The second and third conditions (2.14) fix m and λ. It is imagined that the quantities appearing in the left-hand sides of (2.14) are determined by some experimental observation at the energy scale μ. Since $\left.\Gamma^{(2)}[p]\right|_{p^2=\mu^2}$ and $\left.\Gamma^{(4)}[p_1, p_2, p_3]\right|_S$ depend only on the unknowns m and λ, the normalization conditions (2.14) fix m and λ and therefore the theory.

The theories containing infinitely many couplings need infinitely many normalization conditions. Let us focus on the normalization conditions associated with the two-point functions, which generalize (2.13) and the first of (2.14). When the propagator contains higher derivatives,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n(\partial_\mu\varphi)\Box^n(\partial_\mu\varphi) + \text{vertices},$$

the normalization conditions for $\Gamma^{(2)}[p]$ read

$$\left.\left(\frac{\partial^2}{\partial p^2}\right)^k \Gamma^{(2)}[p]\right|_{p^2=\mu^2} = \sum_{n=k-1}^{\infty}\alpha_n(-1)^n\frac{(n+1)!}{(n-k)!}\mu^{2(n+1-k)}, \tag{2.15}$$

$$\left.\Gamma^{(2)}[p]\right|_{p^2=\mu^2} = \mu^2 + m^2 + \sum_{n=1}^{\infty}\alpha_n(-1)^n\mu^{2n+2}.$$

for $k = 1, 2, \ldots$ and $\alpha_0 = 1$. The need for normalization conditions with $k > 1$ reveals the presence of ghosts.

A necessary condition for unitarity is that no normalization conditions (2.15) with $k > 1$ are necessary to determine the theory. The theorem proven in this paper ensures this, namely that the couplings α_n are inessential and the most general form of a non-renormalizable lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)^2 + \frac{m^2}{2}\varphi^2 + \text{vertices}.$$

The normalization conditions for $\Gamma^{(2)}[p]$ are just

$$\left.\frac{\partial^2\Gamma^{(2)}[p]}{\partial p^2}\right|_{p^2=\mu^2} = 1, \qquad \left.\Gamma^{(2)}[p]\right|_{p^2=\mu^2} = \mu^2 + m^2.$$

The result is trivial for scalar fields and fermions (see below), which I consider just for illustrative purposes. The conclusions apply to the most general theory of spin-0, -1/2, -1, -3/2 and -2 fields in arbitrary dimension greater than two, in the absence of a cosmological constant and when the space-time manifold admits a metric of constant curvature. A generalization that does not need these restrictions exists, and will be discussed later.

3 Removal of divergences and higher derivatives

The purpose of this section and the next one is to show that

in a non-renormalizable quantum field theory of fields of spin 0, 1/2, 1, 3/2 and 2 with no cosmological term, higher-derivative quadratic terms are not turned on by renormalization, if they are absent at the tree level.

On the contrary, if a theory has a higher-derivative propagator at the classical level, infinitely many new higher-derivative quadratic counterterms, not removable using the field equations, can be generated by renormalization.

I begin recalling some basic facts about the removal of divergences in gauge theories and then derive a sort of uniqueness property for the propagator.

Algorithm for the removal of divergences. As usual, I proceed inductively in the perturbative expansion. At the nth step, the "classical" action $S_n[\Phi, K, \lambda]$ is assumed to contain appropriate counterterms so that the quantum action $\Gamma_n[\Phi, K, \lambda]$ is convergent to the order $\hbar^n$ included. Here Φ denote collectively the fields, K are the BRS sources and the sources coupled to the composite operators, and λ are the coupling constants. The theorem of locality of the counterterms ensures that the order-$\hbar^{n+1}$ divergent part $\Gamma_{n+1 \text{ div}}$ of Γ_n is local, since, by inductive assumption, the subdivergences of the graphs contributing to $\Gamma_{n+1 \text{ div}}$ have been removed.

The algorithm for the removal of divergences is made of two ingredients (see for example [5, 6]): field and source redefinitions ("canonical" transformations) and redefinitions of the coupling constants. At the nth step, the divergent terms $\Gamma_{n+1 \text{ div}}$ can be removed [5, 6] with suitable order-$\hbar^{n+1}$ divergent redefinitions of the fields, sources and coupling constants,

$$\Phi \rightarrow \Phi + \delta_{n+1}\Phi, \qquad K \rightarrow K + \delta_{n+1}K, \qquad \lambda \rightarrow \lambda + \delta_{n+1}\lambda,$$

such that the classical action $S_{n+1}[\Phi, K, \lambda]$

$$S_{n+1}[\Phi, K, \lambda] = S_n[\Phi + \delta_{n+1}\Phi, K + \delta_{n+1}K, \lambda + \delta_{n+1}\lambda] = S_n[\Phi, K, \lambda] - \Gamma_{n+1 \text{ div}} + \mathcal{O}(\hbar^{n+2}).$$

generates a quantum action $\Gamma_{n+1}[\Phi, K, \lambda] = \Gamma_n[\Phi, K, \lambda] - \Gamma_{n+1 \text{ div}} + \mathcal{O}(\hbar^{n+2})$ that is convergent to the order $\hbar^{n+1}$ included. Observe that, since δ_{n+1} is of order $\hbar^{n+1}$, at each inductive step only the first order of the Taylor expansion of $S_n[\Phi + \delta_{n+1}\Phi, K + \delta_{n+1}K, \lambda + \delta_{n+1}\lambda]$ in δ_{n+1} is relevant. The higher orders of the Taylor expansion in δ_{n+1} contribute to $\Gamma_{m \text{ div}}$ at the subsequent steps of the iterative procedure, i.e. for $m > n + 1$. Therefore, in the arguments of the next sections we will be concerned only with the first order of the Taylor expansion in the field redefinitions.

The terms that cannot be reabsorbed by means of field redefinitions have to be reabsorbed by means of redefinitions of the coupling constants. When the classical action S_0 does not contain the necessary coupling constants λ, new coupling constants have to be introduced. If the theory is not renormalizable, the divergences can be removed only at the price of introducing infinitely many coupling constants.

The field redefinitions can be of two types: "gauge-covariant" field redefinitions, which do not change the BRS transformations of the fields, and field redefinitions that change the BRS transformations (usually, in a very complicated way). The second type of field redefinitions greatly complicate our discussion and although they can be dealt with using the general renormalization algorithm of [5, 6], they can be avoided using the background field method [7], which is indeed quite popular in quantum gravity (see for example [4]). The background field method is equivalent to the choice of a specific class of gauge fixings. General gauge invariance can be proved using the approach of [5, 6], and the physical results are of course the same.

We are mostly concerned with counterterms proportional to the field equations, because they can be inductively removed by means of field redefinitions. It is sufficient to isolate the first (quadratic) contributions in the quantum fluctuations to the field equations, that is to say the terms proportional to $(-\Box + m^2)\,\varphi$ for scalars, the terms proportional to $(\not{D} + m)\,\psi$ for fermions, the terms proportional to $D_\mu F_{\mu\nu}$ for vectors and the terms proportional to the Ricci tensor and the Ricci curvature for gravity. We have to show that every higher-derivative divergent term that is quadratic in the quantum fluctuations multiplies an inessential coupling, i.e. it can be removed with a vertex counterterm, plus a field redefinition.

Formally, the divergences $\Gamma_{\text{div}}[\varphi, \lambda]$ of the functional integral

$$\int \mathrm{d}[\varphi] \, \exp\left(-S[\varphi, \lambda] + \int J\varphi,\right)$$

where λ denote the parameters of the theory (couplings and masses) are inductively subtracted redefining the fields and the parameters,

$$\int \mathrm{d}[\varphi] \, \exp\left(-S[\varphi, \lambda] + \Gamma_{\text{div}}[\varphi, \lambda] + \int J\varphi\right) = \int \mathrm{d}[\varphi] \, \exp\left(-S[\widetilde{\varphi}(\varphi), \widetilde{\lambda}] + \int J\varphi\right) = \text{finite},$$

and $\widetilde{\varphi}$ is a complicated function of φ, generically containing infinitely many terms. $\widetilde{\varphi}$ plays the role of the bare field, while φ is the renormalized field. The result is that $S[\widetilde{\varphi}(\varphi), \widetilde{\lambda}]$ keeps the same form as $S[\varphi, \lambda]$, that is to say no independent higher-derivative quadratic term is turned on, if it is absent in $S[\varphi, \lambda]$.

Truncations. At the practical level, the renormalization procedure needs to be equipped with the definition of appropriate truncations of the theory. Indeed, since the number of parameters is infinite, infinitely many Feynman diagrams contribute at each order of the $\hbar$ expansion. In general, infinitely many diagrams cannot be computed in one shot. To avoid this difficulty, it is convenient to define truncated theories $\Gamma^{(N)}$. The quantization of a non-renormalizable theory proceeds from the low to the high energies, and is an expansion in $E\kappa$, where E is the energy scale of a physical process and κ is a constant of dimension -1 in units of mass. The truncated theory $\Gamma^{(N)}$ is defined as the theory where only the powers $(E\kappa)^m$ with $m \leq N$ are kept. In the truncated theory finitely many lagrangian terms and coupling constants need to be considered. Moreover, at each order of the $\hbar$ expansion finitely many Feynman diagrams contribute and the divergent terms of order $(E\kappa)^m$ with $m > N$ need not be removed.

In summary, we have two expansions contemporarily: the expansion in powers of $\hbar$ and the expansion in powers of the energy. The fundamental theory is the limit $N \to \infty$ of $\Gamma^{(N)}$. A property of the fundamental theory is a property of $\Gamma^{(N)}$ for every N, not depending on N. For example, the property that we are going to prove (the absence of higher derivatives in the propagators) is a property of the fundamental theory, since it is a property of $\Gamma^{(N)}$ for every N. On the other hand, the dependence on finitely many parameters is not a property of the fundamental theory, because the number of parameters of $\Gamma^{(N)}$ is finite, but depends on N and becomes arbitrarily large when N becomes large.

For concreteness, in pure four-dimensional quantum gravity without a cosmological constant κ is the inverse Planck mass and the action of the $2N$th truncated theory reads symbolically

$$S_{2N} = \frac{1}{\kappa^2} \int \sqrt{g} \left(-R + \kappa^4 \lambda_3 R^3 + \cdots + \kappa^{2N-2} \lambda_N R^N \right).$$

Higher powers of R can be neglected in the $2N$th truncation. Phenomena at energies above the Planck mass can be described (at least in principle) only after appropriate resummations.

Finally, in the presence of a cosmological constant Λ, there is no Λ-independent definition of the "typical energy" E of a physical process and the $2N$th truncated theory is defined as the theory where only the powers $(E\kappa)^p \left(\Lambda\kappa^2\right)^q$ with $p + 2q \leq 2N$ are kept.

"Uniqueness" of the propagator. The theorem we want to prove can be illustrated with a simple argument, which is almost a proof in itself. Let us symbolically write the action S and the field equations $\mathrm{E}_\Phi = 0$ as

$$S = -\frac{1}{2} \int \Phi Q_0 \Phi + \mathcal{O}(\Phi^3), \qquad \mathrm{E}_\Phi = \frac{\delta S}{\delta \Phi} = -Q_0 \Phi + \mathcal{O}(\Phi^2).$$

Here Φ denote the fields and Q_0 is the inverse propagator, which is two-derivative for bosons, and one-derivative for fermions. We can focus on gauge fields, since the statement is trivial in the case of scalars and spin-1/2 fermions (see below). We assume that the gauge fields are expanded around trivial vacuum configurations ($A_\mu = 0$ for vectors, $\psi_\mu = 0$ for spin-3/2 fields and $g_{\mu\nu} = \delta_{\mu\nu}$ for the metric tensor). Gauge invariance to the lowest order in the expansion around the vacua reads $\delta_0 A_\mu = \partial_\mu \Lambda$, $\delta_0 \psi_\mu^\alpha = \partial_\mu \epsilon^\alpha$ and $\delta_0 \phi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, where α is a Lorentz index and $\phi_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$. We have $\delta_0 Q_0 \Phi = 0$.

The higher-derivative quadratic counterterms, which we write as

$$\Delta \mathcal{L}_i = -\frac{1}{2} \Phi Q_i \Phi,$$

where Q_i is some polynomial in the derivatives, have to be invariant under the lowest-order gauge transformations. Using the gauge invariance, it is possible to show that the $\Delta\mathcal{L}_i$s have the form

$$\Delta\mathcal{L}_i = -\frac{1}{2}\Phi Q_0 \widetilde{Q}_i Q_0 \Phi = \frac{1}{2}\Phi Q_0 \widetilde{Q}_i \mathrm{E}_\Phi + \mathcal{O}(\Phi^3),$$

therefore they are proportional to the field equations, up to vertices. This means that the $\Delta\mathcal{L}_i$s can be removed with field redefinitions plus redefinitions of the vertex couplings, without affecting the Φ-propagator. Moreover, since the higher-derivative quadratic counterterms depend on $Q_0\Phi$ and not just Φ, the $\Delta\mathcal{L}_i$s can be promoted to gauge-invariant expressions, with additions of vertices. This ensures that the field redefinitions are BRS covariant.

The meaning of the property that we have just illustrated is a sort of "uniqueness" of the propagator.

In practice, the reason why the quadratic divergent terms are not dangerous is that they are always BRS-exact (because proportional to the field equations) up to vertices. Below, I use these guidelights to prove the theorem in the case of spin 3/2 fields. To generalize the argument when the fields are expanded around non-trivial vacuum configurations, the vacua have to be sufficiently "nice", for example constant, as in the case of scalar vacuum expectation values, or with a constant curvature, as in the case of gravity. On a background with constant curvature the curvature tensors can be expressed in terms of the metric tensor, and the classification of the counterterms becomes relatively simple. Then the proof of the theorem is more or less straightforward. This is shown explicitly in section 7.

Finally, expanding the counterterms around other non-trivial configurations, such as instantons or a space-time with non-constant curvature, the theorem generalizes in the sense that it restricts the form of the lagrangian, but the propagator contains quadratic terms with an arbitrary number of derivatives.

In the next section, I study the most general quantum field theories of fields of spin 0, 1/2, 1 in arbitrary dimensions. In section 5 I study the fields of spin 3/2. The fields of spin 0 and 1/2 can be massive, the fields of spin 1 and 3/2 are assumed to be massless. Gravity without a cosmological constant is studied in the section 6. The theorem is generalized to theories with a cosmological constant in section 7.

I work in the Euclidean framework. In flat space, all space-time indices are low. By convention, I take κ to have the universal dimension -1, in units of mass. For simplicity, I also assume in most cases that the gauge theories are parity invariant. If they are not parity invariant, Chern-Simons terms have to be added in odd dimensions.

4 Fields of spin 0, 1/2 and 1

I begin with the fields of spin 0 and 1/2, which are trivial. The spin-1 case is preparatory for gravity.

Spin 0. The action reads

$$S[\varphi]=\frac{1}{2}\int\varphi\left(-\Box+m^2\right)\varphi+\mathcal{O}(3), \tag{4.1}$$

where $\mathcal{O}(3)$ denote terms that are cubic in the fluctuations around the vacuum configuration. The field equations have the form $\mathrm{E}_\varphi=0$, where

$$\mathrm{E}_\varphi\equiv\left(-\Box+m^2\right)\varphi+\mathcal{O}(2).$$

If the theory contains other fields than φ the terms $\mathcal{O}(2)$ can contain them.

The divergent terms quadratic in φ have the form

$$\sum_{p=1}^{\infty}\kappa^{2p}\sum_{j=0}^{p+1}a_j\ m^{2j}\ \varphi\left(-\Box+m^2\right)^{p+1-j}\varphi=\sum_{p=1}^{\infty}\kappa^{2p}\sum_{j=0}^{p}a_j\ m^{2j}\ \varphi\left(-\Box+m^2\right)^{p-j}\mathrm{E}_\varphi+ \tag{4.2}$$
$$+\sum_{p=1}^{\infty}\kappa^{2p}a_{p+1}\ m^{2p+2}\ \varphi^2+\mathcal{O}(3),$$

where a_j are certain divergent coefficients. A field redefinition

$$\varphi\rightarrow\widetilde{\varphi}(\varphi)=\varphi-\sum_{p=1}^{\infty}\kappa^{2p}\sum_{j=0}^{p}a_j\ m^{2j}\ \left(-\Box+m^2\right)^{p-j}\varphi, \tag{4.3}$$

removes, up to vertices, the quadratic divergent terms of (4.2), but

$$\sum_{p=1}^{\infty}\kappa^{2p}a_{p+1}\ m^{2(p+1)}\ \varphi^2. \tag{4.4}$$

This term can be reabsorbed with a renormalization of the mass.

The argument generalizes immediately to the case when the scalars are coupled to gauge fields, because the covariant derivatives can be commuted up to vertex terms. This is true also for the coupling to gravity, if the vacuum metric is flat. This is the reason why we assume, for the moment, that the theory has no cosmological constant. Note that in the presence of gravity the scalar fields have to be massless, otherwise a cosmological constant is induced by renormalization and the vacuum metric cannot be flat. Under these assumptions, the quadratic divergent terms can be reabsorbed with a gauge-covariant field redefinition, an immediate generalization of (4.3). By the same argument, the $\mathcal{O}(3)$ terms of (4.1) need not contain $\left(-\Box+m^2\right)\varphi$.

Spin 1/2. In the case of the fermion, the lagrangian is

$$\mathcal{L}=\overline{\psi}\left(D\!\!\!\!/\,+m\right)\psi+\mathcal{O}(3).$$

The field equations are $\mathrm{E}_\psi=0$, where

$$\mathrm{E}_\psi\equiv\left(D\!\!\!\!/\,+m\right)\psi+\mathcal{O}(2).$$

We can have only quadratic counterterms of the form

$$I_{n,k}=\overline{\psi}D_{\lambda_1}\cdots D_{\lambda_n}\gamma_{\nu_1}\cdots\gamma_{\nu_k}\psi,\qquad n>1,$$

with variously contracted indices. Proceeding by induction in k, we can assume that no ν_is are contracted together, otherwise we reduce to a case with a lower k. Therefore, all ν_is are contracted with the λ_is and possibly some λ_is are contracted together. Commuting the covariant derivatives, we get, up to vertex terms and terms with a lower k, expressions of the form $\overline{\psi}D\!\!\!\!/^q\psi$, $q>1$. At the κ^pth order the quadratic divergent terms have the form

$$\kappa^p\sum_{j=0}^{p+1}b_j\ m^j\ \overline{\psi}\,(D\!\!\!\!/+m)^{p+1-j}\,\psi=\kappa^p b_{p+1}\ m^{p+1}\ \overline{\psi}\psi+\kappa^p\sum_{j=0}^{p}b_j\ m^j\ \overline{\psi}\,(D\!\!\!\!/+m)^{p-j}\,\mathrm{E}_\psi+\mathcal{O}(3),$$

which can be reabsorbed with the gauge-covariant field redefinition

$$\psi\rightarrow\psi-\frac{1}{2}\sum_{p=1}^{\infty}\kappa^p\sum_{j=0}^{p}b_j\ m^j\ \overline{\psi}\,(D\!\!\!\!/+m)^{p-j}\,\psi,$$

up to vertices, plus a mass renormalization. By the same argument, the $\mathcal{O}(3)$ terms of the lagrangian need not contain $(D\!\!\!\!/+m)\,\psi$ and $\overline{\psi}\left(\overleftarrow{D\!\!\!\!/}+m\right)$.

Spin 1. Consider pure Yang-Mills theory in arbitrary dimension d with lagrangian

$$\mathcal{L}=\frac{1}{\kappa^{d-4}}\left[\ \frac{F_{\mu\nu}^2}{4\alpha}+\sum_{n=1}^{\infty}\lambda_n\kappa^{2n}\Im_n[F,\nabla]\right],\tag{4.5}$$

where $\Im_n[F,\nabla]$ denote collectively the gauge-invariant terms of dimension $2n+4$ in units of mass, constructed with the field strength F and the covariant derivative ∇, with the condition that they are at least cubic in F, up to total derivatives.

The field equations have the form

$$\nabla_\alpha F_{\alpha\beta}=\mathcal{O}(F^2).\tag{4.6}$$

The quadratic counterterms have the form

$$F_{\mu\nu}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}F_{\alpha\beta}\tag{4.7}$$

with variously contracted indices. The derivatives can be interchanged up to cubic terms. If an index λ is contracted with an index of a field strength F in (4.7), such as in

$$\int F_{\mu\nu}\nabla_{\lambda_1}\cdots\nabla_\lambda\cdots\nabla_{\lambda_{2n}}F_{\lambda\beta}\tag{4.8}$$

we first move the covariant derivative ∇_λ till it acts directly on $F_{\lambda\beta}$, then rewrite the counterterm as the sum of a term proportional to the field equations (4.6) plus cubic terms.

If no index λ of (4.7) is contracted with the indices of the field strengths, then the counterterm has the form

$$\int F_{\mu\nu}\Box^p F_{\mu\nu} \tag{4.9}$$

with $p = 0, 1, 2, \ldots$ For $p = 0$ the counterterm is removed with a renormalization of the gauge coupling α. For $p > 0$ we rewrite (4.9), up to vertex counterterms, in the form

$$\int \nabla_\alpha F_{\mu\nu}\Box^{p-1}\nabla_\alpha F_{\mu\nu}$$

and use the Bianchi identity

$$\nabla_\alpha F_{\mu\nu} + \nabla_\nu F_{\alpha\mu} + \nabla_\mu F_{\nu\alpha} = 0, \tag{4.10}$$

to go back to the case (4.8).

The terms proportional to the field equations can be reabsorbed with a BRS covariant field redefinition of the vector potential A_μ, of the form

$$\begin{aligned} A_\mu \rightarrow A_\mu \ &+ c_1\alpha\nabla^\nu F_{\mu\nu} + \alpha^2\left[c_2 F_{\mu\alpha}\nabla_\nu F^{\nu\alpha} + c_3 F^{\nu\alpha}\nabla_\nu F_{\mu\alpha}\right] \\ &+ \alpha^2\left[c_4\nabla_\mu F^2 + c_5\nabla^\alpha\nabla_\alpha\nabla^\nu F_{\mu\nu}\right] + \mathcal{O}(\alpha^3). \end{aligned}$$

Here the c_i are numerical coefficients.

In conclusion, the lagrangian (4.5) preserves its form under renormalization. I will say that (4.5) is "renormalizable", with which I do not mean that it is predictive or renormalizable by power counting, but simply that the missing terms (the higher-derivative terms quadratic in A_μ) are not generated by renormalization.

5 Fields of spin 3/2

In this section I treat the case of spin-3/2 fields, first in four dimensions and then in arbitrary dimension greater than two. I prove that every gauge-invariant quadratic counterterm is necessarily proportional to the field equations up to vertices.

Spin 3/2 in four dimensions. The lagrangian reads

$$\mathcal{L} = i\overline{\psi}_\mu\gamma_5\varepsilon_{\mu\nu\rho\sigma}\partial_\rho\gamma_\sigma\psi_\nu + \text{vertices}.$$

In momentum space, let us define the matrix

$$M_{\mu\nu}(k) = \gamma_5\varepsilon_{\mu\nu\rho\sigma}k_\rho\gamma_\sigma.$$

The field equations read $M_{\mu\nu}(k)\psi_\nu(k) + \mathcal{O}(2) = 0$. Gauge invariance to the lowest order in the quantum fluctuations around the vacuum $\psi_\mu = 0$ ($\delta\psi_\mu = \partial_\mu\epsilon$) is expressed by

$$k_\mu M_{\mu\nu}(k) = 0, \qquad M_{\mu\nu}(k)k_\nu = 0. \tag{5.11}$$

Straightforward calculations prove that

$$\begin{aligned}
M_{\mu\alpha}(k)\left(\delta_{\alpha\beta}-\frac{1}{4}\gamma_\alpha\gamma_\beta\right)M_{\beta\nu}(k) &= k^2\delta_{\mu\nu}-k_\mu k_\nu,\\
-2M_{\mu\alpha}(k)\left(\delta_{\alpha\beta}-\frac{1}{2}\gamma_\alpha\gamma_\beta\right)M_{\beta\nu}(k) &= k^2\gamma_{\mu\nu}-\gamma_{\mu\alpha}k_\alpha k_\nu-k_\mu k_\alpha\gamma_{\alpha\nu},\\
M_{\mu\alpha}(k)\left(\delta_{\alpha\beta}-\frac{1}{4}\gamma_\alpha\gamma_\beta\right)M_{\beta\gamma}(k)M_{\gamma\nu}(k) &= k^2 M_{\mu\nu}(k),\\
M_{\mu\alpha}(k)\not{k}M_{\alpha\nu}(k) &= 2\not{k}\left(k^2\delta_{\mu\nu}-k_\mu k_\nu\right)-k^2M_{\mu\nu}(k),
\end{aligned}\tag{5.12}$$

where $\gamma_{\mu\nu}=[\gamma_\mu,\gamma_\nu]$.

Now, let us consider the structure of the most general quadratic divergent term. We need to distinguish the cases in which the power of κ is even or odd.

If the power of κ is odd, we have a structure

$$\begin{aligned}
&\kappa^{2p+1}\overline{\psi}_\mu(-k)\left(k^2\right)^p\ \Big[(a+b\gamma_5)\left(k^2\delta_{\mu\nu}-k_\mu k_\nu\right)+\\
&\qquad+(c+d\gamma_5)\left(k^2\gamma_{\mu\nu}-\gamma_{\mu\alpha}k_\alpha k_\nu-k_\mu k_\alpha\gamma_{\alpha\nu}\right)\Big]\ \psi_\nu(k),
\end{aligned}\tag{5.13}$$

with $p\geq 0$. This expression is fixed imposing gauge invariance on the most general element of the Clifford algebra, namely a linear combination of 1, γ_5, γ_μ, $\gamma_\mu\gamma_5$, $\gamma_{\mu\nu}$. Relations (5.12) show that (5.13) is always proportional to the matrix $M_{\mu\nu}(k)$, therefore to the field equations up to vertices.

If the power of κ is even, the most general gauge-invariant structure is

$$\kappa^{2p}\overline{\psi}_\mu(-k)\left(k^2\right)^{p-1}\ \Big[(a+b\gamma_5)\,k^2M_{\mu\nu}(k)+(c+d\gamma_5)\,\not{k}\left(k^2\delta_{\mu\nu}-k_\mu k_\nu\right)\Big]\ \psi_\nu(k),\tag{5.14}$$

with $p\geq 1$, and, using (5.12) again, it is proportional to the matrix $M_{\mu\nu}(k)$, therefore to the field equations up to vertices.

We have just studied gauge-invariance to the lowest order in the expansion around the vacuum, and shown that the quadratic divergent terms are proportional to the free-field equations, up to vertices. This is not enough. We have also to be sure that the quadratic divergent terms can be extended to gauge-invariant expressions. Using (5.12) we see that both (5.13) and (5.14) can be rewritten in the form

$$\overline{\psi}_\mu(-k)M_{\mu\alpha}(k)I_{\alpha\beta}(k)M_{\beta\nu}(k)\psi_\nu(k),$$

for a suitable Lorentz matrix $I_{\alpha\beta}(k)$, polynomial in k. Having factorized one matrix $M_{\mu\nu}$ to the right and one to the left, we can also write these objects in the form

$$\overline{W}_{\mu\alpha}(-k)J_{\alpha\beta}(k)W_{\beta\nu}(k),\tag{5.15}$$

with another Lorentz matrix $J_{\alpha\beta}(k)$, polynomial in k. Here $\overline{W}_{\mu\nu}$ and $W_{\mu\nu}$ are the field strengths of the fields, $\overline{W}_{\mu\nu}=\partial_\mu\overline{\psi}_\nu-\partial_\nu\overline{\psi}_\mu$ and $W_{\mu\nu}=\partial_\mu\psi_\nu-\partial_\nu\psi_\mu$. Finally, (5.15) proves that in the higher-derivative quadratic divergent terms the fields $\overline{\psi}_\mu$ and ψ_ν never appear separately from their field strengths. At this point, expression (5.15) can be easily extended to a supergravity scalar.

Spin 3/2 in higher dimensions. The lagrangian reads

$$\mathcal{L} = i\overline{\psi}_\mu \gamma_{\mu\rho\nu}\partial_\rho\psi_\nu + \text{vertices},$$

where $\gamma_{\mu\rho\nu}$ denotes the completely antisymmetrized product of gamma matrices, $\gamma_{\mu\rho\nu} = \gamma_\mu\gamma_\rho\gamma_\nu/6$ + antisymmetrizations. In momentum space, let us define the matrix

$$M_{\mu\nu}(k) = \gamma_{\mu\rho\nu}k_\rho.$$

Gauge invariance to the lowest order is expressed again by (5.11). The higher-dimensional generalizations of (5.12) read

$$\begin{aligned}
\frac{1}{36}M_{\mu\alpha}(k)\left(\delta_{\alpha\beta} - \frac{d-3}{(d-2)^2}\gamma_\alpha\gamma_\beta\right)M_{\beta\nu}(k) &= k^2\delta_{\mu\nu} - k_\mu k_\nu, \\
-\frac{1}{18}M_{\mu\alpha}(k)\left(\delta_{\alpha\beta} - \frac{1}{d-2}\gamma_\alpha\gamma_\beta\right)M_{\beta\nu}(k) &= k^2\gamma_{\mu\nu} - \gamma_{\mu\alpha}k_\alpha k_\nu - k_\mu k_\alpha\gamma_{\alpha\nu}, \\
\frac{1}{36}M_{\mu\alpha}(k)\left(\delta_{\alpha\beta} - \frac{d-3}{(d-2)^2}\gamma_\alpha\gamma_\beta\right)M_{\beta\gamma}(k)M_{\gamma\nu}(k) &= k^2 M_{\mu\nu}(k), \\
M_{\mu\alpha}(k)\not{k}M_{\alpha\nu}(k) &= (d-2)\not{k}\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) - (d-3)k^2 M_{\mu\nu}(k).
\end{aligned} \tag{5.16}$$

A basis for the Clifford algebra is $\gamma_{\mu_1\cdots\mu_i}$ for $i = 0, \cdots, d$, where d is the space-time dimension. The structure of the most general quadratic divergent term is

$$\kappa^{j-1}\overline{\psi}_\mu(-k)\gamma_{\mu_1\cdots\mu_i}k_{\alpha_1}\cdots k_{\alpha_j}\psi_\nu(k), \qquad \kappa^{j-1}\overline{\psi}_\mu(-k)\gamma_{\mu_1\cdots\mu_i}k_{\alpha_1}\cdots k_{\alpha_j}\varepsilon_{\nu_1\cdots\nu_n}\psi_\nu(k),$$

with variously contracted indices. Gauge invariance has not been imposed, yet.

The indices of $\gamma_{\mu_1\cdots\mu_i}$ cannot be contracted among themselves, by antisymmetry, nor with more than one k. In the first class of terms (no ε tensor), we easily find, after imposing gauge invariance, the same terms as in (5.13) and (5.14), with no γ_5. To study the second class of terms (one ε tensor), we use

$$\gamma_{\mu_1\cdots\mu_i} = (-1)^{(n-i)(n-i+1)/2}\,\frac{1}{(n-i)!}\varepsilon_{\mu_1\cdots\mu_i\alpha_{i+1}\cdots\alpha_n}\gamma_{\alpha_{i+1}\cdots\alpha_n}\gamma_5.$$

We get two ε-tensors and a $\gamma_5 = \frac{1}{n!}\varepsilon_{\mu_1\cdots\mu_n}\gamma_{\mu_1\cdots\mu_n}$. Replacing the two εs with a product of Kronecker deltas, we get the same class of terms (5.13) and (5.14), this time multiplied by γ_5 (equal to unity in odd dimensions). From this point on, the discussion proceeds as in the four-dimensional case.

6 Quantum gravity without a cosmological term

In this section I discuss pure quantum gravity without a cosmological term. The metric tensor is expanded around flat space, $g_{\mu\nu} = \delta_{\mu\nu} + \phi_{\mu\nu}$. For the time being, I use the dimensional-regularization technique, which is sensitive to the logarithmic divergences, but ignores the power-like divergences (linear, quadratic, etc.). More general regularization techniques will be discussed later. The l^{th}-loop counterterms in d dimensions are monomials of dimension

$$2+l(d-2), \tag{6.17}$$

constructed with the covariant derivatives of the curvature tensor, multiplied by $\kappa^{(l-1)(d-2)}$.

I want to prove the renormalizability of the theory with lagrangian

$$\mathcal{L}=\frac{1}{\kappa^{d-2}}\sqrt{g}\left[-\ R+\sum_{n=1}^{\infty}\lambda_n\kappa^{n(d-2)}\Im_n[R,\nabla]\right]. \tag{6.18}$$

Here $\Im_n[R,\nabla]$ collectively denote the gauge invariant terms of dimension $n(d-2)+2$ that can be constructed with three or more Riemann tensors $R_{\mu\nu\rho\sigma}$ and the covariant derivative ∇, up to total derivatives. For example, $\Im_1[R,\nabla]$ in six dimensions is a linear combination of terms of the form $R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\delta}R_{\varepsilon\zeta\eta\xi}$, with all possible contractions of indices, but does not contain the terms $R_{\mu\nu\rho\sigma}\nabla_\alpha\nabla_\beta R_{\varepsilon\zeta\eta\xi}$, which would affect the graviton propagator with higher derivatives. If d is odd only the even ns contribute to (6.18). The Ricci tensor $R_{\mu\nu}$ and the scalar curvature R need not appear explicitly in $\Im_n[R,\nabla]$, since they can be removed by means of field redefinitions.

The field equations of (6.18) have the form $\mathrm{E}_{\mu\nu}=0$, with

$$\mathrm{E}_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\mathcal{O}\left(R^2\right), \tag{6.19}$$

so that

$$R_{\mu\nu}=\mathrm{E}_{\mu\nu}-\frac{1}{d-2}g_{\mu\nu}g^{\alpha\beta}\mathrm{E}_{\alpha\beta}+\mathcal{O}\left(R^2\right),\qquad R=-\frac{2}{d-2}g^{\mu\nu}\mathrm{E}_{\mu\nu}+\mathcal{O}\left(R^2\right). \tag{6.20}$$

We have to prove that the quadratic counterterms are proportional to the Ricci tensor or the scalar curvature, so that using (6.19) they can be traded for terms proportional to $\mathrm{E}_{\mu\nu}$, which can be removed by means of covariant field redefinitions, plus terms cubic in the curvature tensors, which can be removed renormalizing the couplings λ_n in (6.18).

The quadratic counterterms that do not trivially contain the Ricci tensor and the Ricci curvature have the form

$$\int\sqrt{g}\ R_{\alpha\beta\gamma\delta}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}R_{\mu\nu\lambda\rho}, \tag{6.21}$$

with variously contracted indices. The derivatives can be freely interchanged, because the difference between two terms (6.21) with interchanged derivatives is a vertex counterterm. If an index λ is contracted with an index of a Riemann tensor $R_{\mu\nu\lambda\rho}$ in (6.21),

$$\int\sqrt{g}\ R_{\alpha\beta\gamma\delta}\nabla_{\lambda_1}\cdots\nabla^{\lambda}\cdots\nabla_{\lambda_{2n}}R_{\mu\nu\lambda\rho} \tag{6.22}$$

we move the covariant derivative ∇_λ till it acts directly on $R_{\mu\nu\lambda\rho}$ and then use the contracted Bianchi identity

$$\nabla^\lambda R_{\mu\nu\lambda\rho}=\nabla_\mu R_{\nu\rho}-\nabla_\nu R_{\mu\rho}. \tag{6.23}$$

This produces terms proportional to $\mathrm{E}_{\mu\nu}$, plus cubic terms in the curvature tensors. It remains to consider the quadratic counterterms of the form

$$\int \sqrt{g}\ R_{\mu\nu\rho\sigma}\Box^p R^{\mu\nu\rho\sigma} \tag{6.24}$$

with $p = 0, 1, 2, \ldots$ Let us first take $p > 0$. Up to vertex counterterms, we can replace (6.24) by

$$\int \sqrt{g}\ \nabla_\alpha R_{\mu\nu\rho\sigma}\Box^{p-1}\nabla^\alpha R^{\mu\nu\rho\sigma}. \tag{6.25}$$

Using the uncontracted Bianchi identity

$$\nabla_\alpha R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\alpha\mu\rho\sigma} + \nabla_\mu R_{\nu\alpha\rho\sigma} = 0, \tag{6.26}$$

we go back to the case (6.22).

We now consider the case $p = 0$ in (6.24). From (6.17) we see that this case is relevant only in four ($d = 4$, $l = 1$) and three ($d = 3$, $l = 2$) dimensions. In three dimensions the Riemann tensor is proportional to the Ricci tensor, since the Weyl tensor is identically zero. In four dimensions the combination

$$\int \sqrt{g}\mathrm{G} = \int \sqrt{g}\ \left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \tag{6.27}$$

is proportional to the Euler characteristic, which is zero at the perturbative level.

We conclude that in arbitrary space-time dimension greater than two every quadratic counterterm is proportional to the Ricci tensor or the Ricci curvature, up to total derivatives and vertex counterterms, and can be removed with a covariant field redefinition of the form

$$g_{\mu\nu} \to g_{\mu\nu} + \sum_{n=1}^{\infty} a_n \kappa^{n(d-2)}\nabla^{n(d-2)-2}R,$$

plus renormalizations of the couplings λ_n. Therefore the lagrangian (6.18) is renormalizable.

If we do not want to use the dimensional-regularization technique, but prefer for example a conventional cut-off regularization, then the argument can be generalized if the renormalized cosmological constant is still set to zero. If there are no masses and parameters with positive dimension in units of mass, this is always possible. If we do not want to set the renormalized cosmological constant to zero, then we have to apply the results of the next section.

In dimension $d > 4$ the term (6.27) can appear among the divergences, multiplied by a power of the cut-off $\overline{\Lambda}$:

$$\overline{\Lambda}^{d-4} f(\overline{\Lambda}\kappa^2)\sqrt{g}\mathrm{G}$$

This term is not present in (6.18), but we can write a renormalizable generalization of (6.18),

$$\mathcal{L} = \frac{1}{\kappa^{d-2}}\sqrt{g}\left[\ -R + \lambda\kappa^2\ \mathrm{G} + \sum_{n=1}^{\infty} \lambda'_n \kappa^{2n+2}\Im'_n[R, \nabla]\right], \tag{6.28}$$

suitable for a regularization technique that is sensitive to the powers of the cut-off $\overline{\Lambda}$, in a subtraction scheme where the renormalized cosmological constant vanishes. In (6.28) $\Im_n'[R,\nabla]$ denote the gauge-invariant terms of dimension $2n+4$ that can be constructed with three or more curvature tensors and the covariant derivative, up to total derivatives. I have remarked in the introduction that (6.27) is a vertex term in every space-time dimension, since it is at least cubic in the quantum fluctuation $\phi_{\mu\nu}$. This proves that the graviton propagator associated with the lagrangian (6.28) does not contain higher derivatives. In particular, the field equations preserve the form (6.19), since the variation of $\int \sqrt{g}\mathrm{G}$ with respect to the metric is $\mathcal{O}(R^2)$.

If there are no masses and parameters with positive dimension in units of mass we can always choose a subtraction scheme where the renormalized coupling constant λ in front of G is set to zero. In this case, we recover the results of the conventional dimensional-regularization scheme.

Obviously, the renormalizability of (6.18) and (6.28) does not depend on the expansion $g_{\mu\nu} = \delta_{\mu\nu} + \phi_{\mu\nu}$. However, it is only with respect to this expansion that higher derivatives do not appear in the graviton propagator. If, for example, we expand the metric in (6.18) or (6.28) around an instanton background, then the graviton propagator does contain higher derivatives.

The results proved so far generalize immediately to theories containing massless fields of spins 0, 1/2, 1, 3/2 and 2 coupled together, since the quadratic part of the action is just the sum of the quadratic parts of the fields. The fields have to be massless in order to be allowed to set the renormalized cosmological constant to zero.

7 Quantum gravity with a cosmological term

A cosmological term is always induced by renormalization when massive fields are coupled to gravity, actually whenever the classical lagrangian contains a dimensionful parameter with positive dimension in units of mass. Even choosing a subtraction scheme where the quartic and quadratic divergences are ignored by default, the masses are responsible for the appearance of logarithmic divergences of the form

$$\frac{m^d}{\varepsilon^n} f(m\kappa)\sqrt{g},$$

which can be removed only with a redefinition of the cosmological constant Λ. It is therefore mandatory to generalize the theorem of the previous sections to a non-vanishing Λ.

I start from classical gravity in d dimensions with lagrangian

$$\mathcal{L}_0 = -\frac{1}{\kappa^{d-2}}\sqrt{g}\ \ (R-\Lambda)\,. \tag{7.29}$$

At the quantum level, the lagrangian $\mathcal{L}_0$ and its field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{\Lambda}{2}g_{\mu\nu} = 0, \tag{7.30}$$

are modified by the additions of infinitely many terms, necessary to renormalize the divergences. The generic gravitational counterterm reads

$$\int \sqrt{g}\ \left(\nabla_{\mu_1}\cdots\nabla_{\mu_{p_1}}\ R^{\alpha_1}{}_{\beta_1\gamma_1\delta_1}\right)\cdots\ \left(\nabla_{\mu_1}\cdots\nabla_{\mu_{pm}}\ R^{\alpha_m}{}_{\beta_m\gamma_m\delta_m}\right) \tag{7.31}$$

with variously contracted indices.

Choice of the background. We have to choose an appropriate gravitational vacuum $\overline{g}_{\mu\nu}$ to define the quantum fluctuations $h_{\mu\nu}$,

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}.$$

The expansion of the Riemann tensor is $R^{\mu}{}_{\nu\rho\sigma} = \overline{R}^{\mu}{}_{\nu\rho\sigma} + R^{(1)\ \mu}{}_{\nu\rho\sigma} + R^{(2)\ \mu}{}_{\nu\rho\sigma} + \mathcal{O}(h^3)$, with

$$R^{(1)\ \mu}{}_{\nu\rho\sigma} = \frac{1}{2}\left(\overline{\nabla}_{\rho}\overline{\nabla}_{\sigma}h^{\mu}_{\nu} + \overline{\nabla}_{\rho}\overline{\nabla}_{\nu}h^{\mu}_{\sigma} - \overline{\nabla}_{\rho}\overline{\nabla}^{\mu}h_{\nu\sigma} - \overline{\nabla}_{\sigma}\overline{\nabla}_{\rho}h^{\mu}_{\nu} - \overline{\nabla}_{\sigma}\overline{\nabla}_{\nu}h^{\mu}_{\rho} + \overline{\nabla}_{\sigma}\overline{\nabla}^{\mu}h_{\nu\rho}\right). \tag{7.32}$$

We do not need the explicit expression of $R^{(2)\ \mu}{}_{\nu\rho\sigma}$, which can be found in [1].

I first choose a vacuum metric $\overline{g}_{\mu\nu}$ satisfying the field equations (7.30) of the lagrangian (7.29),

$$\overline{R}_{\mu\nu} = \frac{\Lambda}{d-2}\overline{g}_{\mu\nu}, \tag{7.33}$$

and study the counterterms (7.31) to the second order in h. Later I show that the vacuum metric is "stable" under renormalization, namely the additional couplings do not affect $\overline{g}_{\mu\nu}$, but at most renormalize the values of the cosmological constant and the Newton constant.

Now, the counterterms (7.31) contain contributions quadratic in h with arbitrary higher derivatives, such as

$$\overline{\nabla}^{m_1}\overline{R}\cdots\overline{\nabla}^{m_n}\overline{R}\ \overline{\nabla}^{p}h_{\mu\nu}\ \overline{\nabla}^{q}h_{\rho\sigma}. \tag{7.34}$$

If the Riemann tensor $\overline{R}_{\mu\nu\rho\sigma}$ does not have a prescribed form, the propagator of the graviton contains arbitrarily many higher derivatives.

To have control on the counterterms, we may assume that the spacetime manifold admits a metric with constant curvature and choose boundary conditions such that the unique solution to (7.33) satisfies also

$$\overline{R}_{\mu\nu\rho\sigma} = \frac{\Lambda}{(d-1)(d-2)}\left(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} - \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}\right). \tag{7.35}$$

In this case, the quadratic terms (7.34) simplify enormously and we can generalize the proof of the previous section. The goal is to show that on the spaces with constant curvature the higher-derivative quadratic terms can be removed by means of covariant field redefinitions and vertex renormalizations.

The metric of a space of constant curvature can always be written in the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_a\ \mathrm{d}x^a}{\left(1+\frac{\Lambda}{4(d-1)(d-2)}x_a x^a\right)^2},$$

in a suitable coordinate frame. Here the indices are raised and lowered with the metric $\breve{g}_{\mu\nu} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_d)$, and $\varepsilon_i = \pm 1$ as appropriate.

Moreover, a Riemannian space has constant curvature if and only if it (locally) admits an a group G_r of motions, with $r = d(d+1)/2$ and if and only if it (locally) admits an isotropy group H_s of $s = d(d-1)/2$ parameters at each point [8].

The restriction (7.35) allows us to work on a class of interesting spaces, such as de Sitter ($\Lambda > 0$) and anti de Sitter ($\Lambda < 0$). The spaces (7.35) are the most symmetric spaces with a cosmological constant, natural candidates for the perturbative vacuum of quantum gravity with a cosmological constant. They are precisely the conformally flat spaces satisfying (7.33). Indeed, it is immediate to prove that

a conformally flat space satisfying (7.33) has constant curvature, i.e. it satisfies also (7.35); conversely, a space of constant curvature is conformally flat and satisfies (7.33).

I define

$$\widehat{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{\Lambda}{(d-1)\,(d-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right),$$

which is more convenient to study the h-expansion, since $\widehat{R}_{\mu\nu\rho\sigma}$ is $\mathcal{O}(h)$.

Inductive hypothesis and strategy of the proof. I assume, by inductive hypothesis, that the complete lagrangian $\mathcal{L}$ is such that its $\mathcal{O}(h)$- and $\mathcal{O}(h^2)$- contributions come only from $\mathcal{L}_0$. The most general expression satisfying these conditions is

$$\mathcal{L} = \frac{1}{\kappa^{d-2}}\sqrt{g}\left[-R + \Lambda + \lambda\kappa^2\ \widehat{\mathrm{G}} + \sum_{n=1}^{\infty}\lambda_n\kappa^{2n+2}\Im_n[\widehat{R}, \nabla, \Lambda]\right] \tag{7.36}$$

and the goal is to show that this lagrangian is renormalizable.

The object $\Im_n[\widehat{R}, \nabla, \Lambda]$ collectively denotes the gauge-invariant terms of dimension $2n+4$ that can be constructed with three or more tensors $\widehat{R}_{\mu\nu\rho\sigma}$, the covariant derivative ∇, and powers of the cosmological constant Λ, up to total derivatives. So, for example, $\Im_1[\widehat{R}, \nabla, \Lambda]$ is a linear combination of terms of the form $\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}_{\alpha\beta\gamma\delta}\widehat{R}_{\varepsilon\zeta\eta\xi}$ with all possible contractions of indices, but does not contain the terms $\widehat{R}_{\mu\nu\rho\sigma}\nabla_\alpha\nabla_\beta\widehat{R}_{\varepsilon\zeta\eta\xi}$, which would affect the h-propagator with higher derivatives. The contracted tensor $\widehat{R}_{\mu\nu}$ and the scalar $\widehat{R}$ need not appear explicitly in $\Im_n[\widehat{R}, \nabla, \Lambda]$, because they can be removed by means of field redefinitions.

The object $\widehat{\mathrm{G}}$ is an appropriate generalization of the G of (6.27) and reads

$$\begin{aligned}\widehat{\mathrm{G}} &= \widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\mu\nu\rho\sigma} - 4\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \widehat{R}^2 + \frac{4(d-3)}{(d-1)(d-2)}\Lambda\,(R-\Lambda)\\ &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 - \frac{(d-3)(d-4)}{(d-1)(d-2)}\Lambda\,(2R-\Lambda)\,.\end{aligned}$$

$\widehat{\mathrm{G}}$ is constructed so that the spacetime integral of $\sqrt{g}\ \widehat{\mathrm{G}}$ does not contain h-quadratic terms with higher derivatives, when it is expanded around the background (7.35). Precisely, after a straightforward, but lengthy calculation, using (7.32), we find

$$\int\sqrt{g}\ \widehat{\mathrm{G}} = \frac{8(d-3)}{(d-1)(d-2)^2}\Lambda^2\int\sqrt{\overline{g}} + \mathcal{O}\left(h^3\right). \tag{7.37}$$

This identity is a nontrivial fact and the key ingredient to prove our theorem.

The $2N$th truncated theory is the theory with lagrangian

$$\mathcal{L}_{2N} = \frac{1}{\kappa^{d-2}}\sqrt{g}\left[-R+\Lambda+\lambda\kappa^2\ \widehat{\mathrm{G}}+\sum_{n=1}^{N-2}\lambda_n\kappa^{2n+2}\Im_n[\widehat{R},\nabla,\Lambda]\right],$$

where only the powers $(E\kappa)^p\left(\Lambda\kappa^2\right)^q$ with $p+2q\leq 2N$ are kept, E denoting the reference energy scale. It is easy to see that at each order of the $\hbar$ expansion we need to consider only a finite number of Feynman diagrams in the truncated theory. Indeed, the powers of Λ are bounded by the very same definition of the truncation and, on dimensional grounds, the terms of $\Im_n[\widehat{R},\nabla,\Lambda]$ can be multiplied only by a polynomial in $\lambda_n\kappa^{2n}$, with $n\leq N-2$.

Field equations. Due to (7.37), the terms of $\mathcal{L}$ quadratic in h come only from $\mathcal{L}_0$, so the field equations of $\mathcal{L}$ are $\mathrm{E}_{\mu\nu}=0$ with

$$\begin{aligned}\mathrm{E}_{\mu\nu} &= \kappa^2\frac{\delta S[\overline{g}+h]}{\delta h^{\mu\nu}} = -\frac{1}{2}\overline{\nabla}^2 h_{\mu\nu}+\frac{1}{2}\overline{g}_{\mu\nu}\overline{\nabla}^2 h-\frac{1}{2}\overline{g}_{\mu\nu}\overline{\nabla}_\alpha\overline{\nabla}_\beta h^{\alpha\beta}-\frac{1}{2}\overline{\nabla}_\mu\overline{\nabla}_\nu h \\ &\quad +\frac{1}{2}\overline{\nabla}_\mu\overline{\nabla}^\alpha h_{\alpha\nu}+\frac{1}{2}\overline{\nabla}_\nu\overline{\nabla}^\alpha h_{\mu\alpha}+\frac{\Lambda}{(d-1)(d-2)}h_{\mu\nu}+\frac{\Lambda(d-3)}{2(d-1)(d-2)}\overline{g}_{\mu\nu}h+\mathcal{O}\left(h^2\right) \\ &= R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\left(R-\Lambda\right)+\mathcal{O}\left(h^2\right)=\widehat{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\widehat{R}+\mathcal{O}\left(h^2\right). \qquad (7.38)\end{aligned}$$

where $h=h_{\mu\nu}\overline{g}^{\mu\nu}$ and indices are lowered and raised using $\overline{g}_{\mu\nu}$. We have

$$\widehat{R}_{\mu\nu}=\mathrm{E}_{\mu\nu}-\frac{1}{d-2}g_{\mu\nu}g^{\alpha\beta}\mathrm{E}_{\alpha\beta}+\mathcal{O}\left(h^2\right),\qquad \widehat{R}=-\frac{2}{d-2}g^{\mu\nu}\mathrm{E}_{\mu\nu}+\mathcal{O}\left(h^2\right). \qquad (7.39)$$

We see that under the assumption that the lagrangian has the form (7.36), the field equations contain no higher-derivative terms linear in h and therefore the form of the h-propagator is the standard one. Moreover, the field equations (7.39) are trivially solved by $h=0$. This means that the background $\overline{g}_{\mu\nu}$ is not affected by the infinitely many couplings λ,λ_n. The background metric is sensitive only to the value of the cosmological constant, which is renormalized by radiative corrections, but otherwise the form of the vacuum metric is stable under renormalization.

To proceed with the proof of our theorem, we need a more refined expression for the field equations than (7.38) and (7.39), that is to say

$$\mathrm{E}_{\mu\nu}=\widehat{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\widehat{R}+\mathcal{O}\left(\widehat{R}^2\right), \qquad (7.40)$$

so that

$$\widehat{R}_{\mu\nu}=\mathrm{E}_{\mu\nu}-\frac{1}{d-2}g_{\mu\nu}g^{\alpha\beta}\mathrm{E}_{\alpha\beta}+\mathcal{O}\left(\widehat{R}^2\right),\qquad \widehat{R}=-\frac{2}{d-2}g^{\mu\nu}\mathrm{E}_{\mu\nu}+\mathcal{O}\left(\widehat{R}^2\right). \qquad (7.41)$$

To prove (7.41) I start from the most general expression of the field equations, which is, symbolically,

$$\mathrm{E}_{\mu\nu}=\widehat{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\widehat{R}+\sum_{k\geq 1}\sum_{\{p_i\}}\prod_{i=1}^{k}\left(\nabla^{p_i}\widehat{R}_i\right). \qquad (7.42)$$

The last term of this formula collects the contributions the terms $\widehat{\mathrm{G}}$ and $\Im_n[\widehat{R},\nabla,\Lambda]$ in (7.36). The terms of (7.42) with $k>1$ are $\mathcal{O}\left(\widehat{R}^2\right)$ and we do not need to discuss them. The terms with $k=1$ can come only from $\widehat{\mathrm{G}}$ and have the form

$$b\widehat{R}_{\mu\nu}+cg_{\mu\nu}\widehat{R}+a'\nabla_\mu\nabla_\nu\widehat{R}+b'\Box\widehat{R}_{\mu\nu}+c'g_{\mu\nu}\Box\widehat{R} \tag{7.43}$$

up to terms with a higher number of tensors $\widehat{R}$. These can be included in the terms of (7.42) with $k>1$. Due to (7.38), we know that (7.43) has to be $\mathcal{O}\left(h^2\right)$. Using (7.32), we have

$$\widehat{R}_{\mu\nu}=-\frac{1}{2}\overline{\nabla}_\mu\overline{\nabla}_\nu h+\frac{1}{2}\overline{\nabla}_\mu\overline{\nabla}^\alpha h_{\alpha\nu}+\frac{1}{2}\overline{\nabla}_\nu\overline{\nabla}^\alpha h_{\mu\alpha}-\frac{1}{2}\overline{\nabla}^2 h_{\mu\nu}+\frac{\Lambda(h_{\mu\nu}-h\overline{g}_{\mu\nu})}{(d-1)(d-2)}+\mathcal{O}\left(h^2\right),$$
$$\widehat{R}=-\overline{\nabla}^2 h+\overline{\nabla}^\alpha\overline{\nabla}^\beta h_{\alpha\beta}-\frac{\Lambda}{d-2}h+\mathcal{O}\left(h^2\right).$$

We see that no combination (7.43) can be $\mathcal{O}\left(h^2\right)$ and therefore the terms with $k=1$ are absent in (7.42) and the field equations of (7.36) have the form

$$\mathrm{E}_{\mu\nu}=\widehat{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\widehat{R}+\sum_{k>1}\sum_{\{p_i\}}\prod_{i=1}^{k}\nabla^{p_i}\widehat{R}_i, \tag{7.44}$$

wherefrom (7.40) and (7.41) follow.

The result is completely general, that is to say it does not depend on the expansion $g_{\mu\nu}=\overline{g}_{\mu\nu}+h_{\mu\nu}$ around a background with constant curvature, although we used this expansion to prove (7.40). Equation (7.40) can be proved more directly differentiating $\int\sqrt{g}\widehat{\mathrm{G}}$ with respect to the metric.

Now we analyse the most general counterterms (7.31) and prove that all of them can be reabsorbed with covariant redefinitions of the metric tensor, of the form

$$g_{\mu\nu}\rightarrow g_{\mu\nu}+\mathcal{O}\left(\widehat{R}\right),\qquad \text{i.e.}\qquad h_{\mu\nu}\rightarrow h_{\mu\nu}+\mathcal{O}\left(h\right), \tag{7.45}$$

plus renormalizations of the cosmological constant, the Newton constant and the parameters λ,λ_n appearing in (7.36), thus preserving the structure (7.36) to all orders in perturbation theory.

Analysis of the counterterms. It is convenient to rewrite the counterterms in the basis $\widehat{R}_{\mu\nu\rho\sigma}$,

$$\int\sqrt{g}\ \left(\nabla_{\mu_1}\cdots\nabla_{\mu_{p_1}}\ \widehat{R}^{\alpha_1}{}_{\beta_1\gamma_1\delta_1}\right)\cdots\ \left(\nabla_{\mu_1}\cdots\nabla_{\mu_{pm}}\ \widehat{R}^{\alpha_m}{}_{\beta_m\gamma_m\delta_m}\right) \tag{7.46}$$

The terms (7.46) with $m\geq 3$ are of the type $\Im_n[\widehat{R},\nabla,\Lambda]$ and are renormalized by means of redefinitions of the couplings λ_n, $n\geq 1$. We need to study only the terms with $m=0,1,2$:

$$I_1=\int\sqrt{g}\ ,\qquad\qquad I_2=\int\sqrt{g}\ \nabla_\mu\cdots\nabla_{\mu_p}\ \widehat{R}^{\alpha}{}_{\beta\gamma\delta}\ ,$$
$$I_3=\int\sqrt{g}\ \ \widehat{R}^{\alpha_1}{}_{\beta_1\gamma_1\delta_1}\nabla_{\mu_1}\cdots\nabla_{\mu_p}\ \widehat{R}^{\alpha_2}{}_{\beta_2\gamma_2\delta_2}\ .$$

Terms I_1 and I_2. The term I_1 is a renormalization of the cosmological constant. The term I_2 vanishes for $p > 0$. For $p = 0$ all contractions of indices in I_2 give

$$\int \sqrt{g}\ \widehat{R} = \int \sqrt{g}\ \left(R - \frac{d}{d-2}\Lambda\right), \tag{7.47}$$

which can be reabsorbed with renormalizations of the Newton constant and the cosmological constant.

Terms I_3 containing $\widehat{R}_{\mu\nu}$ and $\widehat{R}$. Using (7.41), the terms I_3 containing $\widehat{R}_{\mu\nu}$ and $\widehat{R}$ are proportional to the field equations up to higher-order terms in $\widehat{R}$. We have objects of the form

$$\int \sqrt{g}\ \widehat{R}_{\alpha\beta\gamma\delta}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}\widehat{R}_{\mu\nu} = \int \sqrt{g}\ \widehat{R}_{\alpha\beta\gamma\delta}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}\mathrm{E}_{\mu\nu} + \mathcal{O}\left(\widehat{R}^3\right),$$

with all possible contractions of indices. Divergences of this form can be reabsorbed with a redefinition of the metric tensor of the form

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}\widehat{R}_{\alpha\beta\gamma\delta}, \qquad \text{i.e.} \qquad h_{\mu\nu} \to h_{\mu\nu} + \nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}\widehat{R}_{\alpha\beta\gamma\delta},$$

plus renormalizations of the couplings λ_n, $n \geq 1$.

Terms I_3 not containing $\widehat{R}_{\mu\nu}$ and $\widehat{R}$. The terms I_3 that do not trivially contain $\widehat{R}_{\mu\nu}$ and $\widehat{R}$ have the form

$$\int \sqrt{g}\ \widehat{R}_{\alpha\beta\gamma\delta}\nabla_{\lambda_1}\cdots\nabla_{\lambda_p}\widehat{R}_{\mu\nu\lambda\rho}, \tag{7.48}$$

with variously contracted indices.

i) The terms (7.48) with $p > 0$ can be treated as the terms (6.21) studied in the absence of a cosmological constant. First, let us observe that the Bianchi identity (6.26) holds also with hatted tensors:

$$\nabla_\alpha\widehat{R}_{\mu\nu\rho\sigma} + \nabla_\nu\widehat{R}_{\alpha\mu\rho\sigma} + \nabla_\mu\widehat{R}_{\nu\alpha\rho\sigma} = 0. \tag{7.49}$$

When covariant derivatives are commuted in (7.48), we get terms with a higher number of tensors $\widehat{R}$ plus terms of the form (7.48) with a lower value of p, but belonging to the same truncation $\Gamma^{(N)}$ of the theory:

$$\begin{aligned}[\nabla_\alpha, \nabla_\beta]\, T_{\mu_1\cdots\mu_n} = &-\sum_{i=1}^{n} \widehat{R}^{\rho}{}_{\mu_i\alpha\beta}\, T_{\mu_1\cdots\mu_{i-1}\rho\mu_{i+1}\cdots\mu_n} \\ &-\frac{\Lambda}{(d-1)(d-2)}\sum_{i=1}^{n}\left(T_{\mu_1\cdots\mu_{i-1}\alpha\mu_{i+1}\cdots\mu_n} g_{\mu_i\beta} - T_{\mu_1\cdots\mu_{i-1}\beta\mu_{i+1}\cdots\mu_n} g_{\mu_i\alpha}\right).\end{aligned} \tag{7.50}$$

If in (7.48) an index of the covariant derivatives is contracted with an index of a tensor $\widehat{R}_{\mu\nu\lambda\rho}$, we have terms of the form

$$\int \sqrt{g}\ \widehat{R}_{\alpha\beta\gamma\delta}\nabla_{\lambda_1}\cdots\nabla^{\lambda}\cdots\nabla_{\lambda_p}\widehat{R}_{\mu\nu\lambda\rho}. \tag{7.51}$$

Here we commute the covariant derivatives till ∇_λ acts directly on $\widehat{R}_{\mu\nu\lambda\rho}$ and then use the Bianchi identity, obtaining terms proportional to the field equations plus higher-order terms in $\widehat{R}$, plus terms (7.48) with lower values of p belonging to the same truncation level.

If in (7.48) no index of the covariant derivatives is contracted with an index of a tensor $\widehat{R}_{\mu\nu\lambda\rho}$, we have terms of the form

$$\int \sqrt{g}\ \widehat{R}_{\mu\nu\rho\sigma}\Box^p\widehat{R}^{\mu\nu\rho\sigma}$$

with $p = 1, 2, \ldots$ these objects can be rewritten, after commuting the derivatives a sufficient number of times, as

$$\int \sqrt{g}\ \nabla_\alpha\widehat{R}_{\mu\nu\rho\sigma}\Box^{p-1}\nabla^\alpha\widehat{R}^{\mu\nu\rho\sigma},$$

plus higher orders in $\widehat{R}$ and terms of the form (7.48) with lower values of p. Using the Bianchi identity we go back to the case (7.51).

Proceeding inductively in p, we can lower the value of p arbitrarily, remaining in the same truncation. In the end, we need to discuss only the $p = 0$ terms.

ii) The terms (7.48) with $p = 0$ are

$$\int \sqrt{g}\ \widehat{R}_{\alpha\beta\gamma\delta}\widehat{R}_{\mu\nu\lambda\rho}.$$

The possible contractions of indices give

$$\int \sqrt{g}\ \widehat{R}^{\mu\nu\lambda\rho}\widehat{R}_{\mu\nu\lambda\rho}, \qquad \int \sqrt{g}\ \widehat{R}^{\mu\nu}\widehat{R}_{\mu\nu}, \qquad \int \sqrt{g}\ \widehat{R}^2. \tag{7.52}$$

The second and third term of this list are proportional to the field equations $\mathrm{E}_{\mu\nu}$ up to higher orders in $\widehat{R}$. The first term of (7.52) is $\int \sqrt{g}\widehat{\mathrm{G}}$ plus a term I_1, a term I_2, the second and third terms of (7.52).

Concluding, up to higher orders in $\widehat{R}$, the first term of (7.52) can be removed with renormalizations of λ, the cosmological constant and the Newton constant, plus a covariant redefinition of the metric tensor.

We have therefore proven that the structure (7.36) is preserved to every order and the h-propagator does not contain higher derivatives. For each truncation $\Gamma^{(N)}$ of the theory the removal of the divergences requires a finite number of steps at each order of the $\hbar$ expansion.

Background independence. Going through the proof of the theorem, we see the the key-ingredients, which are (7.49), (7.50) and (7.41), do not depend on the metric $\overline{g}_{\mu\nu}$ around which the expansion is performed. The most general version of our theorem is:

the action

$$S = \frac{1}{\kappa^{d-2}}\int \sqrt{g}\left[-R + \Lambda + \lambda\kappa^2\ \widehat{\mathrm{G}} + \sum_{n=1}^{\infty}\lambda_n\kappa^{2n+2}\Im_n[\widehat{R}, \nabla, \Lambda]\right] \tag{7.53}$$

is renormalizable with redefinitions of the cosmological constant, the Newton constant and the parameters λ, λ_n, plus a covariant redefinition of the metric tensor.

The choice of a vacuum metric $\overline{g}_{\mu\nu}$ with constant curvature is relevant only to the form of the graviton propagator. If the vacuum metric satisfies (7.35), the graviton propagator of (7.53) does not contain higher derivatives. If the vacuum metric does not satisfy (7.35), then the graviton propagator of (7.53) does contain higher derivatives.

Coupling to matter. Finally, the theorem generalizes straightforwardly to the case when the fields of spin 0, 1/2, 1, 3/2 and 2 are coupled together. It is sufficient to recall that by spin conservation the mixed quadratic terms, e.g. the spin-1/spin-2 quadratic terms

$$\int \sqrt{g}\ F_{\alpha\beta}\nabla_{\lambda_1}\cdots\nabla_{\lambda_p}\widehat{R}_{\mu\nu\lambda\rho} \tag{7.54}$$

are actually vertices up to terms proportional to the field equations. For example, in the case of (7.54) the covariant derivatives can be commuted up to vertices and terms with a lower value of p belonging to the same truncation of the theory. If the index of a covariant derivative is contracted with an index of F or $\widehat{R}$, then, commuting the covariant derivatives a sufficient number of times and using the Bianchi identities, we obtain terms proportional to the field equations plus vertices. So, we can assume that the indices of the covariant derivatives are contracted among themselves. This leads to objects of the form

$$\int \sqrt{g}\ F_{\alpha\beta}\Box^p\widehat{R}_{\mu\nu\lambda\rho},$$

which admit no non-trivial contraction.

8 Perturbative vacuum and the vacuum of quantum gravity

In this section I make some illustrative comments on the perturbative and quantum vacua of the theory defined by the lagrangian (7.53). A more detailed analysis of perturbation theory will be presented in a separate paper.

The metric of constant curvature is such that the graviton propagator is two-derivative. If the space-time manifold admits a metric of constant curvature, we choose boundary conditions such that the unique solution to the Einstein equations $\widehat{R}_{\mu\upsilon} - g_{\mu\nu}\widehat{R}/2 = 0$ is precisely that metric. Then the metric of constant curvature is also the unique solution to the complete field equations $\mathrm{E}_{\mu\nu} = 0$ of the action (7.53), where $\mathrm{E}_{\mu\nu}$ is given by (7.40) and (7.44). This means that the metric of constant curvature is an extremal of (7.53).

It is reasonable to expect that the quantum corrections, which certainly change the vacuum metric in the bulk of the space-time manifold, do not affect the behavior of the metric at the boundary of the space-time manifold, although the value of the asymptotic curvature gets renormalized. So, if the perturbative-vacuum metric is of constant curvature, the vacuum of quantum gravity, i.e. the minimum of the quantum action $\Gamma[g_{\mu\nu}]$, has an asymptotically constant curvature.

However, the metric of constant curvature is not a minimum of the action (7.53). The second derivative of (7.53), calculated on the vacuum metric $\overline{g}_{\mu\nu}$ of constant curvature, can be written explicitly, because we have shown that the quadratic terms of (7.53) receive contributions only from the Einstein and cosmological terms. We have

$$\frac{1}{2}\int h_{\mu\nu}\left.\frac{\delta^2 S}{\delta g_{\mu\nu}\delta g_{\rho\sigma}}\right|_{g_{\alpha\beta}=\overline{g}_{\alpha\beta}} h_{\rho\sigma} = \frac{1}{4\kappa^{d-2}}\int\sqrt{\overline{g}}\ \mathrm{H}[\widetilde{h},\overline{g}],$$

$$\mathrm{H}[\widetilde{h},\overline{g}] = \left(\overline{\nabla}_\mu\widetilde{h}_{\nu\rho}\right)^2 - \frac{1}{d-2}\left(\overline{\nabla}_\mu\widetilde{h}\right)^2 - 2\left(\overline{\nabla}^\mu\widetilde{h}_{\mu\nu}\right)^2 + \frac{2\Lambda}{(d-1)(d-2)}\left(\widetilde{h}_{\mu\nu}^2 + \frac{1}{d-2}\widetilde{h}^2\right),$$

where $\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \overline{g}_{\mu\nu}h/2$. $\mathrm{H}[h,\overline{g}]$ is not positive definite, due to the known negative-definite contribution of the conformal factor. For example, $h_{\mu\nu} = h\overline{g}_{\mu\nu}/d$ gives

$$\mathrm{H}[\widetilde{h},\overline{g}] = -4\frac{d-1}{d^2(d-2)}\left(\overline{\nabla}_\mu\widetilde{h}\right)^2 + 2\Lambda\widetilde{h}^2\frac{d-1}{d^2(d-2)^2}.$$

This means that either the total action (7.53) is not bounded from below (in the Euclidean framework) or, if it is bounded from below, its absolute minimum is not the metric of constant curvature. In the latter case different boundary conditions select the true perturbative vacuum and the quantum vacuum need not have an asymptotically constant curvature.

On the other hand, the perturbative calculations (propagator, vertices and Feynman diagrams) of the theory (7.53) expanded around a metric of constant curvature appear to be well-defined, at least if the curvature is negative (anti-de Sitter space) [9]. When the curvature is positive the Green functions behave less nicely [10]. In particular, the perturbative calculations around anti-de Sitter space do not exhibit the typical difficulties of the expansion around a “wrong” configuration. For comparison, let us take a scalar-field theory with potential $V(\phi) = -\mu^2\phi^2/2 + g^2\phi^4/4$. A perturbative expansion around the local maximum $\phi = 0$ is problematic, because the propagator is tachyonic. In principle, the physical results should not depend on the configuration $\overline{\phi}$ around which the expansion in performed, but the boundary conditions are crucial to select the appropriate $\overline{\phi}$. Precisely, two expansions around configurations $\overline{\phi}_1$ and $\overline{\phi}_2$ satisfying the same boundary conditions are equivalent (after appropriate resummations). Since the local maximum of $V(\phi)$ tends to zero at infinity, while the minimun tends to $v = \mu/g$, we have no reason to expect that the expansion around $\overline{\phi} = v$ is equivalent to the expansion around $\overline{\phi} = 0$, even admitting that we can make sense of the it (e.g. with an analytical continuation from negative μ^2).

Thus, the best criterion to choose the perturbative vacuum in a non-renormalizable theory is not necessarily the minimum of the classical action. Ultimately, it is not even necessary that the classical action, which has no direct physical significance, be bounded from below. The first check is to see whether the perturbative expansion around the temptative vacuum is well-defined or not. Therefore, the metric of negative constant curvature is a good candidate for the right perturbative vacuum of quantum gravity, even if it is not the absolute minimum of the action (7.53). Then, we are allowed to argue that vacuum of quantum gravity has a negative asymptotically constant curvature.

9 Conclusions

The results of this paper suggest that the theories with infinitely many couplings can be studied in a perturbative sense also at high energies. Various questions are well-posed and can be answered. In particular, a whole class of lagrangian terms is not turned on by renormalization, if it is absent at the tree level. For example, the lagrangian of quantum gravity in arbitrary space-time dimension greater than two can be conveniently organized as the sum

$$\mathcal{L} = \frac{1}{\kappa^{d-2}}\sqrt{g}\left[-R+\Lambda+\lambda\kappa^2\ \widehat{\mathrm{G}}+\sum_{n=1}^{\infty}\lambda_n\kappa^{2n+2}\Im_n[\nabla,\widehat{R},\Lambda]\right], \tag{9.55}$$

where $\widehat{\mathrm{G}}$ and $\Im_n[\nabla,\widehat{R},\Lambda]$ are defined in the paper. The lagrangian (9.55) is "renormalizable", namely preserves its form under renormalization.

If the theory has no cosmological constant or the space-time manifold admits a metric of constant curvature, the propagators of the fields are not affected by higher derivatives. The metric of constant curvature is an extremal, but neither an absolute minimum, nor a local minimum of the action (9.55). Therefore, either the action (9.55) is not bounded from below or there exists a different perturbative minimum. Some considerations suggest that the metric of negative constant curvature is the right perturbative vacuum to formulate the perturbative expansion of the theory (9.55), even if it is not a minimum of the classical action. Then we can argue that the quantum vacuum has a negative asymptotically constant curvature.

It would be interesting to see if the theorem proven here can be further generalized to treat the vertices containing higher derivatives of the metric.

The results of this paper might also be useful in effective field theory, for a more convenient treatment of the radiative corrections to the low-energy limit, and to properly address the cosmological constant problem.

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